Fano profiles in scattering around exceptional points

Lukas Schwarz\textsuperscript{1}, Holger Cartarius\textsuperscript{1}, Günter Wunner\textsuperscript{1}, Walter Dieter Heiss\textsuperscript{2,3}, and Jörg Main\textsuperscript{1}

\textsuperscript{1} Institut für Theoretische Physik, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany
\textsuperscript{2} Department of Physics, University of Stellenbosch, 7602 Matieland, South Africa
\textsuperscript{3} National Institute for Theoretical Physics (NITheP), Western Cape, South Africa

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Abstract. In a previous paper it has been shown that the interference of the first and second order pole of the Green’s function at an exceptional point, as well as the interference of the first order poles in the vicinity of the exceptional point, gives rise to asymmetric scattering cross section profiles. In the present paper we demonstrate that these line profiles are indeed well described by the Beutler-Fano formula, and thus are genuine Fano resonances. Also further away from the exceptional points excellent agreement can be found by introducing energy dependent Fano parameters.

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1 Introduction

Exceptional points [1] are points in the physical parameter space of non-Hermitian operators where eigenvalues and eigenstates exhibit specific properties (cf. [2–4]): Both eigenvalues and eigenfunctions coalesce, the eigenvalues undergo a branch point singularity, the eigenstates inter-change and acquire a geometrical phase when encircling the exceptional point. Furthermore, at an exceptional point the operator cannot be diagonalised but assumes the Jordan normal form; for the wave functions this entails a linear (or higher order) term in the time evolution besides the usual exponential behaviour. Non-Hermitian operators appear in different branches of physics [3], in particular when the interaction of systems with an environment is described by effective Hamiltonians or operators. Examples include open quantum systems with complex Hamiltonians [5], optical systems with loss or gain [6–12], microcavities [13], electronic circuits [14], and laser physics [15], to quote just a few.

As has been pointed out in the literature [4,16–20] another striking feature of exceptional points is the Green’s function which possesses a pole of second order in addition to the usual first order pole. Using the simple model of two coupled damped harmonic oscillators [21,22] to mimic the general behaviour of scattering systems near exceptional points, Heiss and Wunner [23] (hereafter referred to as paper I) have shown that the interference of the first and second order poles at the exceptional point as well as the interference of the two first order poles in close vicinity of the exceptional point give rise to asymmetric line profiles in scattering cross sections. It is the purpose of the present paper to demonstrate that both at and in the vicinity of the exceptional point the cross sections indeed give rise to Fano profiles locally, and therefore can be interpreted as genuine Fano resonances. Moreover, the cross sections can globally be described by the Beutler-Fano formula, if energy-dependent Fano parameters are introduced.

2 The model

For the reader’s convenience we briefly review the model introduced in paper I. The system consists of two one-dimensional harmonic oscillators with unperturbed eigenfrequencies $\omega_1$, $\omega_2$ and damping constants $k_1$, $k_2$, coupled by a spring with spring constant $f$ and damping constant $g$, and periodically driven by an external force with frequency $\omega$. After setting up the equations of motion in phase space time-periodic solutions with real frequency $\omega$ are found by the roots of the characteristic polynomial of the matrix

$$M = \begin{pmatrix} -2g - 2k_1 & 2g & -f - \omega_1^2 & f \\ 2g & -2g - 2k_2 & f & -f - \omega_2^2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\det(M + i\omega) = 0,$$  \hspace{1cm} (1)

and exceptional points occur for complex values of $\omega$ where the first derivative of the characteristic polynomial vanishes,

$$\frac{d}{d\omega} \det[M + i\omega] = 0.$$ \hspace{1cm} (2)

Equations (1), (2) lead to 4 real-valued equations for the 8 real parameters $\omega_1, \omega_2, k_1, k_2, g, f, R(\omega), \Im(\omega)$. If we provide specific values for the eigenfrequencies and damping...
constants, we are left with 4 equations for the 4 unknowns $g, f, \Re(\omega), \Im(\omega)$, which in general have to be solved numerically. In doing so the physical constraints on the spring and coupling constants $f > 0$, $g > 0$, and the frequency $\Re(\omega) > 0$ and width $\Im(\omega) < 0$ have to be imposed.

For the special case that the individual oscillators are undamped, $k_1 = k_2 = 0$, which will be used in the following, the exceptional points can be calculated in closed analytical form:

\begin{align}
\Re[\omega_{EP}] &= \frac{1}{2\sqrt{2}} \sqrt{\left(3\omega_1^2 + \omega_2^2\right) \left(\omega_1^2 + 3\omega_2^2\right)}, \\
\Im[\omega_{EP}] &= \frac{-1}{2\sqrt{2}} \frac{|\omega_1^2 - \omega_2^2|}{\omega_1^2 + \omega_2^2}, \\
g_{EP} &= -\Im[\omega_{EP}] = \frac{1}{2\sqrt{2}} \frac{|\omega_1^2 - \omega_2^2|}{\omega_1^2 + \omega_2^2}, \\
f_{EP} &= \frac{(\omega_1^2 - \omega_2^2)^2}{4(\omega_1^2 + \omega_2^2)}. 
\end{align}

3 Generic behaviour near an exceptional point

In close vicinity of an exceptional point the higher dimensional eigenvalue problem can be reduced to an effective two-channel scattering problem with the effective Hamiltonian (see appendix)

\begin{equation}
H(f) = H_0 + fV = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix} + f \begin{pmatrix} \epsilon_1 & \delta_1 \\ \delta_2 & \epsilon_2 \end{pmatrix}. 
\end{equation}

The quantities $\Omega_1, \Omega_2, \epsilon_1, \epsilon_2, \delta_1, \delta_2$ appearing in the matrices are directly related to the original parameters of the system. We emphasise that the eigenvalue trajectories of the two coalescing states in the original and the reduced problem coincide near the exceptional point (cf. paper I).

The advantage of the reduction to the form (4) is the analytic availability of the Green’s function and thus the scattering amplitudes for both, (i) near the exceptional point:

\begin{align}
G(E) &= (E - H(f_{EP}))^{-1} \\
&= \frac{1}{E_1 - E_2 - E - E_1} \begin{pmatrix} E_1 - \Omega_2 - \epsilon_2 f & \delta_1 f \\ \delta_2 f & E_1 - \Omega_1 - \epsilon_1 f \end{pmatrix} \\
&= \frac{1}{E_1 - E_2 - E - E_2} \begin{pmatrix} E_2 - \Omega_2 - \epsilon_2 f & \delta_1 f \\ \delta_2 f & E_2 - \Omega_1 - \epsilon_1 f \end{pmatrix}, 
\end{align}

and (ii) at the exceptional point:

\begin{equation}
f_{EP} = \frac{-i(\Omega_1 - \Omega_2)}{i(\epsilon_1 - \epsilon_2) \pm 2\sqrt{\delta_1 \delta_2}}.
\end{equation}

4 Results

4.1 Cross sections

In Fig. 1 we illustrate the behaviour of the cross sections as the two eigenstates approach the exceptional point. We assume no self-damping of the individual oscillators ($k_1 = k_2 = 0$) and choose $\omega_1 = 2.00$ and $\omega_2 = 2.10$. The exceptional point then appears at $f_{EP} = 0.005, \omega_{EP} = 0.0499$, and $\omega_{EP} = 2.0500 - 0.0499i$. The asymmetry of the line profile is evident from Fig. 1. The right-hand column of Fig. 1 resolves the contributions of the individual pole terms and the interference term to the cross section.

Far away from the exceptional point (top panel) each peak in the cross section can be associated with one resonance pole, one with a larger width, and the other with a smaller width. The interference term arising from the two individual terms in (5) is small in comparison with the individual pole terms, yet it forces the vanishing of scattering between the two peaks.

By contrast, closer to the exceptional point (middle panel) neither of the two peaks in the cross section can be associated any more with one of the two resonance poles, and the interference term is seen to have a sizeable effect. In fact, the two poles without interference could not be resolved as they would be overlapping resonances. Furthermore, the interference term not only produces the zero between the peaks but also determines their widths and positions.

The bottom panel shows the situation at the exceptional point. Note that in this case the interference term is given by $2\Re[T_{22}^{(2)}(T_{22}^{(1)})^*-1]$, with $T_{22}^{(1)}$ and $T_{22}^{(2)}$ being the contributions of the first and the second order pole terms, respectively. Here the two poles at $\omega_{EP}$ have equal heights, they are exactly cancelled by the interference term leading again to a zero in the cross section, i.e. zero scattering.

Fig. 1 confirms, in an exemplary way, the asymmetric Fano-Feshbach-like line profiles found in paper I around
exceptional points. To establish that this is no coincidence we recall the Beutler-Fano formula (see e.g. [24])

$$\sigma(\epsilon) = \frac{(\epsilon + q)^2}{\epsilon^2 + 1}. \quad (10)$$

It quite generally describes line profiles for a physical process where a continuum state interacts with a bound state embedded in the continuum. In (10)

$$\epsilon = \frac{E - E_R}{\Gamma/2} \quad (11)$$

is the reduced energy which measures the energy relative to the position of the resonance $E_R$ in units of the half-width $\Gamma/2$ being the width of the quasi-bound state in the continuum (note that the interaction endows the quasi-bound state with a width). The parameter $q$ determines the shape of the resonance and depends on the ratio of the transition matrix elements linking the initial state to the discrete and continuum parts of the final state.

In Fig. 2 we illustrate, for the same physical parameters as in Fig. 1, Fano fits to the cross sections at and near the exceptional point. The fitting parameters are $E_R$, $\Gamma$, and $q$, in addition we have a scaling factor in front of eq. (10) to scale the height of the resonance and a global shift of the cross section to account for a background.

It is evident from Fig. 2 that both at and further away from the exceptional point the Beutler-Fano formula locally describe the asymmetric line profiles perfectly. It thus confirms that they are Fano resonances.

### 4.2 Energy dependent Fano parameters

If one allows for two sets of energy dependent Fano parameters the two peaks in the cross sections and the asymmetric line profile can indeed be modelled over a far wider range of the energy. Examples are shown in Fig. 3. It can be seen that farther away from the exceptional point the profiles are locally well described by two Fano forms. This, however, is not surprising since by their very nature each of the separate resonances, one with a larger and one with a smaller width, should be describable by Fano profiles, with different parameters. It is remarkable to see that the peaks retain Fano profiles even closer to and in particular at the exceptional point, where the interference terms become increasingly important.

The idea of energy dependent Fano parameters has been put forward by Magnuson et al. [20] in their study of overlapping resonances. Starting from the $S$ matrix with two isolated resonances $E_k = E_R - i\Gamma/2$ ($k = 1, 2$), and a smooth reaction background described by a phase shift $\delta$, they show that the cross section can be brought into

### Table 1. Resonance energies $E_R$, widths $\Gamma$, and asymmetry parameter $q$ of the Fano profiles around the exceptional point shown in Fig. 2.

<table>
<thead>
<tr>
<th></th>
<th>$E_R$</th>
<th>$\Gamma$</th>
<th>$q$</th>
</tr>
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<tbody>
<tr>
<td>a)</td>
<td>2.04628</td>
<td>0.00078</td>
<td>-6.43165</td>
</tr>
<tr>
<td>b)</td>
<td>2.04242</td>
<td>0.00289</td>
<td>-3.03946</td>
</tr>
<tr>
<td>c)</td>
<td>2.03899</td>
<td>0.00623</td>
<td>-2.09385</td>
</tr>
<tr>
<td>d)</td>
<td>2.04917</td>
<td>0.02755</td>
<td>-0.06180</td>
</tr>
</tbody>
</table>

### Fig. 1. (a) $f = f_{EP}+1.0$, (b) $f = f_{EP}+0.5$, (c) $f = f_{EP}+0.3$, (d) $f = f_{EP}$. The black dot marks the position of the exceptional point.

### Fig. 2. Fano fits (red) to the cross sections at and near the exceptional point. The physical parameters are the same as in Fig. 1. (a) $f = f_{EP}+1.0$, (b) $f = f_{EP}+0.5$, (c) $f = f_{EP}+0.3$, (d) $f = f_{EP}$. The black dot marks the position of the exceptional point.
Since in our model there is no background, the phase shift is 0, in which case (12) simplifies to

\[ \sigma(E) = \frac{1}{\epsilon_2^2 + 1} \left( \frac{\epsilon_1 + \epsilon_2}{\epsilon_1^2 + 1} \right)^2. \]  

The comparison with (10) shows that \( \epsilon_2 \) formally assumes the role of an energy dependent asymmetry parameter \( q(E) \) for resonance 1, and vice versa.

Apart from the scale factor for the height of the cross section, in the extended ansatz (13) the fitting parameters are \( E_1, \Gamma_1, E_2, \Gamma_2 \).

Examples for fits to the cross sections with energy dependent asymmetry parameters at and in the vicinity of exceptional points are shown in Fig. 4. The agreement between the calculated cross section and the energy dependent Fano fit is found to be perfect over the complete energy range.

![Fig. 3. Fano fit to the cross section at and near the exceptional point with two sets of Fano parameters. The physical parameters are the same as in Fig 1.](image)

![Table 2. Resonance energies, widths, and asymmetry parameters of the two Fano profiles shown in Fig. 3.](table)

<table>
<thead>
<tr>
<th>E, ( \Gamma ), ( q )</th>
<th>E, ( \Gamma ), ( q )</th>
</tr>
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<tbody>
<tr>
<td>a) 2.0463 0.0008 -6.4770 2.5416 0.0995 299.0411</td>
<td></td>
</tr>
<tr>
<td>b) 2.0424 0.0029 -3.1026 2.2997 0.0969 36.6894</td>
<td></td>
</tr>
<tr>
<td>c) 2.0390 0.0062 -2.1014 2.2028 0.0934 12.1206</td>
<td></td>
</tr>
<tr>
<td>d) 2.0244 0.0438 -1.8446 2.0785 0.0440 1.8728</td>
<td></td>
</tr>
</tbody>
</table>

The fitting parameters required for this excellent agreement are listed in Tab. 3. For sufficient distance from the EP (rows d-f), where the interference term plays a minor role, this is to be expected as we encounter essentially two independent and well separated resonances associated with poles the positions of which are given in the table.

![Table 3. Fitting parameters for the cross sections shown in Fig. 4 calculated using eq. (13).](table)

<table>
<thead>
<tr>
<th>E, ( \Gamma ), ( q )</th>
<th>E, ( \Gamma ), ( q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 2.04070 0.10792 2.04990 -0.09239</td>
<td></td>
</tr>
<tr>
<td>b) 2.49559 1.04493 2.50780 -0.91573</td>
<td></td>
</tr>
<tr>
<td>c) 3.47188 3.20257 3.62857 -2.28382</td>
<td></td>
</tr>
<tr>
<td>d) 2.03912 0.01471 2.20748 0.18673</td>
<td></td>
</tr>
<tr>
<td>e) 2.04252 0.06680 2.30187 0.19218</td>
<td></td>
</tr>
<tr>
<td>f) 2.04626 0.00196 2.54172 0.19923</td>
<td></td>
</tr>
</tbody>
</table>

![Fig. 4. Fano fits to cross sections at and in the vicinity of exceptional points using eq. (13). The blue curves are the calculated cross sections, the red curves denote the fits.](image)
In contrast, at the EP (rows a-c) the two peaks cannot be seen as two separated resonances; in fact, here the interference term is crucial to produce the two peaks (see [23]). It therefore comes as no surprise that the best fits produce results - negative widths - that can no longer be interpreted as physical resonances. In fact, the double pole invoked by the EP is different in character from the double pole as considered in [20]: the two-dimensional coefficient matrix of the double pole at the EP does not have full rank but rank unity (see [4]) meaning that the entries are correlated in a particular way. In this context, we also note that our scattering matrices are not unitary as the underlying Hamiltonian is not hermitian.

5 Summary

Starting from the fact that in scattering systems exceptional points give rise to a second order pole in the Green’s function, we have investigated the effect on the shape of the scattering cross section as one approaches the exceptional point. We have shown that it is the interference term of the poles which generates the asymmetric line profiles. In this paper we have demonstrated that the line profiles in the neighbourhood of the exceptional point are to be interpreted as Fano resonances. Moreover, by allowing for energy dependent Fano parameters the cross sections can also globally be described as actual Fano profiles.

We feel that by this analysis we have given a new and deeper understanding of Fano-resonances. On the one side there is the mathematical mechanism of two coalescing eigenvalues - an EP - on the other the physical origin of two interacting near resonances where the one can be a single particle resonance and the other a bound state in the continuum that, owing to the interaction, has acquired itself a width. This is the typical Fano-Feshbach scenario. Whether or not all resonances that can be fitted by the traditional Fano-Feshbach procedure have their origin in the underlying classical problem, the generic behaviour of the poles which generates the asymmetric line profiles.

It should be stressed that, in spite of the simplicity of the underlying classical problem, the generic behaviour of line profiles studied in this paper near exceptional points applies to any physical two-channel scattering or transmission problem. Therefore we did establish the direct link between the appearance of asymmetric Fano profiles and the occurrence of exceptional points in parameter space.

Appendix

To reduce the higher dimensional eigenvalue problem of the complex matrix $M(f)$ to a two-dimensional one in the vicinity of an exceptional point, one first has to determine the left and right eigenvectors of $M$ for $f$ close to the EP, $|E_1(f)|$, $|E_2(f)|$, $\langle E_1(f) | \rangle$, $\langle E_2(f) | \rangle$. Next the matrix (1) is split into an $f$-independent part $M_0$ and a part linear in $f$, $M = M_0 + fM_1$. For the $f$-dependent part the $2 \times 2$ matrix

$$H_{0,ij} = \langle E_i(f) | M_0 | E_j(f) \rangle$$

is set up and diagonalized. This yields eigenvalues $\Omega_1(f)$, $\Omega_2(f)$, and eigenvectors $|\Omega_1(f)\rangle$, $|\Omega_2(f)\rangle$. Finally the $f$-dependent part of $M$ is represented in the basis of these new eigenvectors

$$V_{0,ij} = \langle \Omega_i(f) | M_1 | \Omega_j(f) \rangle,$$

which leads to the reduced problem

$$H(f) = H_0 + fV = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix} + f \begin{pmatrix} \epsilon_1 & \delta_1 \\ \delta_2 & \epsilon_2 \end{pmatrix}.$$ 

References